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LETTER TO THE EDITOR

Bosons and fermions interacting integrably with the Korteweg–de Vries field

B A Kupershmidt

The University of Tennessee Space Institute, Tullahoma, Tn 37388, USA

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Abstract. Two different super-integrable extensions of the Korteweg–de Vries equation and associated systems are discussed.

Every integrable system in the Lax form (Wilson 1979)

$$L_t = [P_+, L], \tag{1}$$

with (matrix) differential operators L and P_+,

$$L = \sum_{i=0}^{n} u_i \xi^i, \qquad \xi \equiv \partial \equiv \partial/\partial x, \tag{2}$$

can be extended out to include additional variables by adding to L in (2) a pseudo-differential tail:

$$L \to \tilde{L} = L + \sum_s \mu_s \xi^{-1} \nu_s^t, \tag{2'}$$

where, for each s, μ_s and ν_s are either both *even* (bosons) or both *odd* (fermions). This procedure provides a canonical way to extend standard integrable differential Lax equations. The extended systems remain integrable: all the flows (1), for various P commuting with L, commute and have an infinite common set of conservation laws, as follows from general facts about Lax equations (Wilson 1979, Kupershmidt 1982b). Analogous extensions exist for discrete Lax equations (Kupershmidt 1982a, b) and for standard super-integrable differential Lax equations (Kupershmidt 1984b).

When L is either self-adjoint or skew-adjoint: $L^\dagger = \pm L$, the variables μ_s and ν_s in (2′) can be specialised such that $\tilde{L}^\dagger = \pm \tilde{L}$ as well, e.g.

$$\nu_s = \mu_s A_s, \qquad A_s^t = \pm A_s, \qquad \nu_s \text{ and } \mu_s \text{ are fermions}, \tag{3f}$$

$$\nu_s = \mu_s A_s, \qquad A_s^t = \mp A_s, \qquad \nu_s \text{ and } \mu_s \text{ are bosons}, \tag{3b}$$

where A_s are constant matrices. Other possibilities for the tail, especially when L is a scalar operator, include

$$\tilde{L}^\dagger = \tilde{L}: \sum \mu_s \xi^{-1} \mu_s^t, \qquad \tilde{L}^\dagger = -\tilde{L}: \sum (\mu_s \xi^{-1} \nu_s^t - \nu_s \xi^{-1} \mu_s^t) \qquad \text{(fermions)}, \tag{4f}$$

$$\tilde{L}^\dagger = \tilde{L}: \sum (\mu_s \xi^{-1} \nu_s^t - \nu_s \xi^{-1} \mu_s^t), \qquad \tilde{L}^\dagger = -\tilde{L}: \sum \mu_s \xi^{-1} \mu_s^t \qquad \text{(bosons)}. \tag{4b}$$

Not much else is known at the present time about super-intregrable systems in general and super-extensions of the standard integrable systems in particular (the word

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'super' refers to the presence of fermions and/or bosons), with the exception of supersymmetric σ-models (D'Auria and Sciuto 1980), supersymmetric Toda lattices (Ol'shanetsky 1983), and supersymmetric KP hierarchy (Manin and Radul 1984). In particular, there is no theoretical information available (outside the usual method of factorising the Lax operator (2')) about the central object in the theories of integrable systems: the Miura map. In this note I shall discuss, in the case of the Korteweg–de Vries (KdV) equation, two practical instances where the Miura map does exist although its origin remains a mystery.

The first case is a slight generalisation of the s-KdV system in Kupershmidt (1984a):

$$\begin{aligned} &u_t = \partial(3u^2 - u_{xx} + 12E_1),\\ &\boldsymbol{\varphi}_t = P_1(\boldsymbol{\varphi}), \qquad \boldsymbol{z}_t = P_1(\boldsymbol{z}), \qquad \boldsymbol{y}_t = P_1(\boldsymbol{y}),\\ &p(u) = p(z_j) = p(y_j) = 0, \qquad p(\varphi_i) = 1 \end{aligned} \tag{5}$$

$$E_1 = E_1[\boldsymbol{\varphi}, \boldsymbol{z}, \boldsymbol{y}] := \boldsymbol{\varphi}^t\boldsymbol{\varphi}_x + 3(\boldsymbol{z}^t\boldsymbol{y}_x - \boldsymbol{y}^t\boldsymbol{z}_x), \qquad P_1 = P_1[u] = 3(u\partial + \partial u) - 4\partial^3, \tag{6}$$

where $p(\sigma) = 0$ or 1 according to whether σ is boson or fermion. The system (5)–(6) has the Lax representation (1) with

$$L = -\xi^2 + u + \boldsymbol{\varphi}^t\xi^{-1}\boldsymbol{\varphi} + \boldsymbol{z}^t\xi^{-1}\boldsymbol{y} - \boldsymbol{y}^t\xi^{-1}\boldsymbol{z}, \qquad P_+ = P_1, \tag{7}$$

so that $\tilde{L}^\dagger = \tilde{L}$ by (4). The first Hamiltonian structure of (5)–(6) is

$$\begin{aligned} &u_t = \partial(\delta H/\delta u), \qquad \varphi_{i,t} = \tfrac{1}{4}\,\delta H/\delta\varphi_i, \qquad z_{j,t} = -\tfrac{1}{12}\,\delta H/\delta y_j,\\ &y_{j,t} = \tfrac{1}{12}\,\delta H/\delta z_j, \end{aligned} \tag{8}$$

$$H = u^3 + \tfrac{1}{2}u_x^2 + 12uE_1 - 8\boldsymbol{\varphi}^t\boldsymbol{\varphi}_{xxx} - 48\boldsymbol{z}^t\boldsymbol{y}_{xxx}. \tag{9}$$

Unless $\boldsymbol{z}$ and $\boldsymbol{y}$ are both absent and $\boldsymbol{\varphi}$ is a scalar, *there is no second Hamiltonian structure* for (5)-(6): thus, even if the Miura map exists it must lose its Hamiltonian property (Kupershmidt and Wilson 1981).

The s-MKdV system corresponding to (5)-(6)

$$\begin{aligned} &v_t = \partial(2v^3 - v_{xx} + 6vE_2 + 3E_{2,x}),\\ &\boldsymbol{\alpha}_t = P_2(\boldsymbol{\alpha}), \qquad \boldsymbol{a}_t = P_2(\boldsymbol{a}), \qquad \boldsymbol{b}_t = P_2(\boldsymbol{b}),\\ &p(v) = p(a_j) = p(b_j) = 0, \qquad p(\alpha_i) = 1, \end{aligned} \tag{10}$$

$$E_2 = E_1[\boldsymbol{\alpha}, \boldsymbol{a}, \boldsymbol{b}], \qquad P_2 = P_1[v^2 - v_x] + 6E_2\partial - 3E_{2,x}, \tag{11}$$

is mapped into the s-KdV system (5)–(6) by the Miura map

$$u = v^2 + v_x + E_2, \qquad \boldsymbol{\varphi} = (\partial + v)(\boldsymbol{\alpha}), \qquad \boldsymbol{z} = (\partial + v)(\boldsymbol{a}), \qquad \boldsymbol{y} = (\partial + v)(\boldsymbol{b}). \tag{12}$$

The system (10)–(11) is Hamiltonian only when $\boldsymbol{a}$ and $\boldsymbol{b}$ are absent and $\boldsymbol{\alpha}$ is a scalar. Since (5)-(6) is Galilean invariant, one uses (12) by the method of Kupershmidt (1982c) to construct a deformation of (5)-(6):

$$\begin{aligned} &U_t = \partial(3U^2 - U_{xx} + 2\varepsilon^2U^3 + 12(1 + 2\varepsilon^2U)E_3 + 12\varepsilon E_{3,x}),\\ &\boldsymbol{\Phi}_t = P_3(\boldsymbol{\Phi}), \qquad \boldsymbol{Z}_t = P_3(\boldsymbol{Z}), \qquad \boldsymbol{Y}_t = P_3(\boldsymbol{Y}),\\ &p(U) = p(Z_j) = p(Y_j) = 0, \qquad p(\Phi_i) = 1, \end{aligned} \tag{13}$$

$$\begin{aligned} &E_3 = E_1[\boldsymbol{\Phi}, \boldsymbol{Y}, \boldsymbol{Z}],\\ &P_3 = P_1[U + \varepsilon^2U^2 - \varepsilon U_x] + 24\varepsilon^2E_3\partial - 12\varepsilon^2E_{3,x}, \end{aligned} \tag{14}$$

together with its contraction onto (5)–(6):

$$u = U + \varepsilon U_x + \varepsilon^2 U^2 + 4\varepsilon^2 E_3,$$
$$\varphi = P_4(\boldsymbol{\Phi}), \qquad z = P_4(\boldsymbol{Z}), \qquad y = P_4(\boldsymbol{Y}), \qquad P_4 := 1 + 2\varepsilon\partial + 2\varepsilon^2 U. \tag{15}$$

Since U is a conservation law (CL) in (13) inverting (15) provides a new construction for an infinity of CLs for (5)–(6), in addition to the standard formula $\{\mathrm{Res}\, L^{k/2},\ k \in Z_+,\ \tilde{L}$ in (7)$\}$.

The second case of the s-KdV system is

$$u_t = \partial(3u^2 - u_{xx} + 3E_4),$$
$$\boldsymbol{\omega}_t = P_5(\boldsymbol{\omega}), \qquad \boldsymbol{\sigma}_t = P_5(\boldsymbol{\sigma}), \qquad \boldsymbol{f}_t = P_5(\boldsymbol{f}), \tag{16}$$
$$p(u) = p(f_i) = 0, \qquad p(\omega_j) = p(\sigma_j) = 1,$$
$$E_4 = E_4[\boldsymbol{\omega}, \boldsymbol{\sigma}, \boldsymbol{f}] = \boldsymbol{\omega}^t\boldsymbol{\sigma} + \boldsymbol{f}^t\boldsymbol{f}, \qquad P_5 = P_5[u] = 6\partial u - \partial^3, \tag{17}$$

with the s-MKdV system

$$v_t = \partial(2v^3 - v_{xx} + 6vE_5),$$
$$\boldsymbol{c}_t = P_6(\boldsymbol{c}), \qquad \boldsymbol{\beta}_t = P_6(\boldsymbol{\beta}), \qquad \boldsymbol{\gamma}_t = P_6(\boldsymbol{\gamma}), \tag{18}$$
$$p(v) = p(c_i) = 0, \qquad p(\beta_j) = p(\gamma_j) = 1,$$
$$E_5 = E_4[\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{c}], \qquad P_6 = P_5[v^2] + 6E_5\partial, \tag{19}$$

and the Miura map

$$u = v_x + v^2 + E_5, \qquad \boldsymbol{\omega} = P_7(\boldsymbol{\beta}), \qquad \boldsymbol{\sigma} = P_7(\boldsymbol{\gamma}),$$
$$\boldsymbol{f} = P_7(\boldsymbol{c}), \qquad P_7 := \partial + 2v. \tag{20}$$

The s-KdV system (16, 17) has the first Hamiltonian structure

$$u_t = \partial(\delta H/\delta u), \qquad f_{i,t} = \partial(\delta H/\delta f_i),$$
$$\omega_{j,t} = -2\partial(\delta H/\delta\sigma_j), \qquad \sigma_{j,t} = 2\partial(\delta H/\delta\omega_j), \tag{21}$$
$$H = u^3 + \tfrac{1}{2}u_x^2 + 3uE_4, \tag{21'}$$

and does not have a second Hamiltonian structure unless both $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$ are absent and $\boldsymbol{f}$ is a scalar, in which case the variables $u \pm f$ decouple (16)–(17) into a pair of non-interacting KdV fields. Again, since (16)–(17) is Galilean invariant, one uses (20) to construct a deformation of (16)–(17):

$$U_t = \partial(3U^2 - U_{xx} + 2\varepsilon^2 U^3 + 3(1 + 2\varepsilon^2 U)E_6),$$
$$\boldsymbol{\Omega}_t = P_8(\boldsymbol{\Omega}), \qquad \boldsymbol{\Sigma}_t = P_8(\boldsymbol{\Sigma}), \qquad \boldsymbol{F}_t = P_8(\boldsymbol{F}), \tag{22}$$
$$p(U) = p(F_i) = 0, \qquad p(\Omega_j) = p(\Sigma_j) = 1,$$
$$E_6 = E_4[\boldsymbol{\Omega}, \boldsymbol{\Sigma}, \boldsymbol{F}], \qquad P_8 = P_5[U + \varepsilon^2 U^2] + 6\varepsilon^2 E_6\partial, \tag{23}$$

together with its contraction $C(\varepsilon)$:

$$u = U + \varepsilon U_x + \varepsilon^2 U^2 + \varepsilon^2 E_6, \qquad \boldsymbol{\omega} = P_9(\boldsymbol{\Omega}),$$
$$\boldsymbol{\sigma} = P_9(\Omega), \qquad \boldsymbol{f} = P_9(\boldsymbol{F}), \qquad P_9 = 1 + \varepsilon\partial + 2\varepsilon^2 U, \tag{24}$$

onto (16)–(17). Since the Lax form of (16)–(17) is not known, the deformation (22)–(23) provides the only available route to construct an infinity of CLs for the s-KdV system (16)–(17): since U in (22) is a CL, one inverts (24) to find $U=\Sigma_{n\geqslant 0}\,\varepsilon^n H_n$. Notice, that in contrast to the deformed s-KdV system (13)–(14), our new deformed system (22)–(23) depends only upon ε^2 while the map $C(\varepsilon)$ (24) depends upon ε. Thus, one obtains an auto-Bäcklund transformation $C(\varepsilon)^{-1}\circ C(-\varepsilon)$ of the s-KdV system (16)–(17) into itself.

In conclusion, the formulae presented here constitute, no doubt, only a tip of the puzzling iceberg and only a small part of the whole story. For example, applying extension (2′) to the operator

$$L=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\xi+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}v$$

of the MKdV hierarchy, and specialising the resulting $\tilde{L}$ to be a skew-adjoint and -1-circulant (Kupershmidt and Wilson 1981), results in a s-MKdV system very different from both (10)–(11) and (18)–(19).

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